AD-A240 317

ATION PAGE

Form Approved
OMB No. 0704-0188

1 AGENCY USE ONLY (Leave blank)	2 REPORT DATE August 1991	3 REPORT TYPE AND DATES COVERED Professional paper

4 TITLE AND SUBTITLE THREE-VALUED LOGICS AND CONDITIONAL EVENT ALGEBRAS	5 FUNDING NUMBERS PR: ZE90 PR: CD32 PE: 0602 6N PE: 0305108K WU: DN30036 WU: DN48828
6 AUTHOR(S) I. R. Goodman	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Ocean Systems Center San Diego, CA 92152-5000	8 PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Office Chief of Naval Research Independent Exploratory Development Programs (IED) OCNR-20T Arlington, VA 22217 Office of the Secretary of Defense Research and Engineering Washington, DC 20363	10 SPONSORING/MONITORING AGENCY REPORT NUMBER
11 SUPPLEMENTARY NOTES	
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.	12b. DISTRIBUTION CODE

DTIC ELECTE SEP 1 3 1991 S C D

13 ABSTRACT (Maximum 200 words)

First, a review of the progress is presented for the development of conditional event algebras. Following this, a new canonical bijection of isomorphisms is derived. This is an extension of the usual indicator function mapping to that between all possible truth-functional three-valued logics and all possible choices of conditional event operators extending unconditional boolean ones. Relations between the conditional event algebra proposed by Goodman & Nguyen and £^{3}, as well as that proposed by Schay, Adams, and Calabrese and Sob$_3$ are derived, among other isomorphic correspondences.

91-10558

Published in *Proceedings of ISUMA '90*, Dec 1990.

14. SUBJECT TERMS data fusion conditional events conditional event algebra	15. NUMBER OF PAGES
	16 PRICE CODE

17 SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19 SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAME AS REPORT

NSN 7540-01-280-5500

91 9 12 165

Standard form 298

21a NAME OF RESPONSIBLE INDIVIDUAL	21b TELEPHONE (include Area Code)	21c OFFICE SYMBOL
I. R. Goodman	(619) 553-4014	Code 421

NSN 7540-01-280-5500

Standard form 298

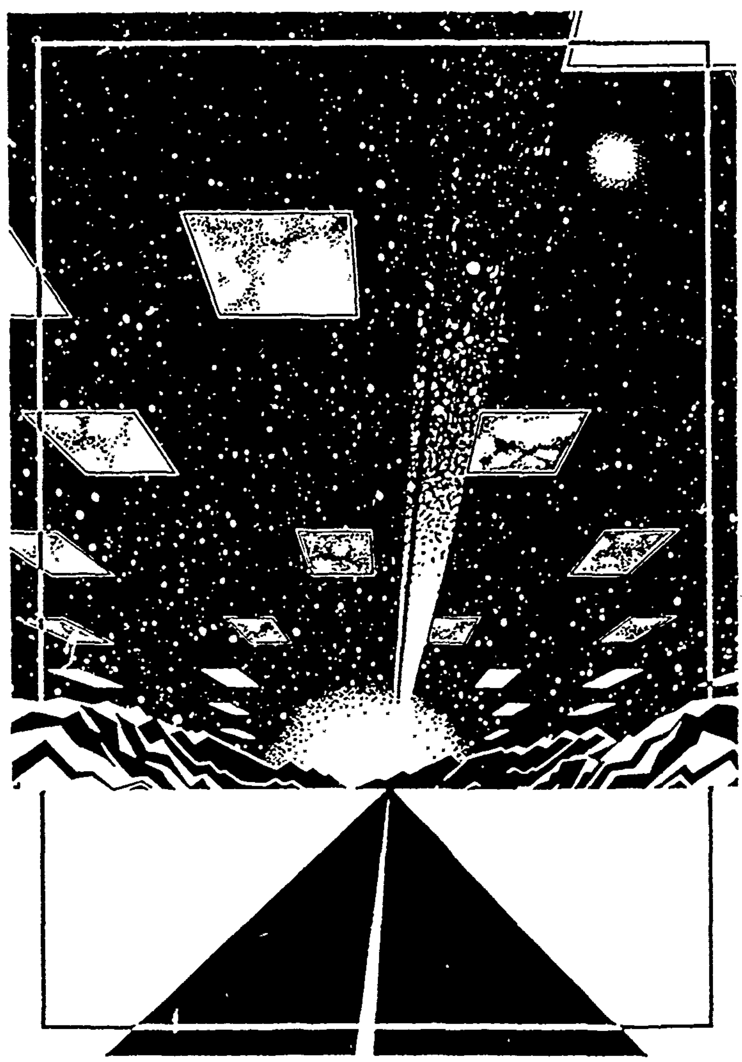
ISUMA '90

Uncertainty Modeling and Analysis

Proceedings of ISUMA '90
The First International Symposium on Uncertainty Modeling and Analysis

University of Maryland
College Park, Maryland, USA
December 3-5, 1990

Edited by
Bilal M. Ayyub



IEEE COMPUTER SOCIETY PRESS

INSTITUTE OF ELECTRICAL AND ELECTRONICS ENGINEERS, INC.

THREE-VALUED LOGICS AND CONDITIONAL EVENT ALGEBRAS

I.R. Goodman

Code 421
Command & Control Department
Naval Ocean Systems Center
San Diego, CA 92152-5000

Abstract

First, a review of the progress is presented for the development of conditional event algebras. Following this, a new canonical bijection of isomorphisms is derived. This is an extension of the usual indicator function mapping to that between all possible truth-functional three-valued logics and all possible choices of conditional event operators extending unconditional boolean ones. Relations between the conditional event algebra proposed by Goodman & Nguyen and $Ł_3$, as well as that proposed by Schay, Adams, and Calabrese and Sob_3 are derived, among other isomorphic correspondences.

Note on General Notation, Conventions

In addition to the usual use of equality =, set inclusion $\subseteq$, set membership ε, class of all subsets of or power class $P(\ )$, null set $\emptyset$, etc., we introduce $\overset{d}{=}$ to mean "is defined to be", emphasizing the difference between provable, as in =, and the former. Throughout, R (as opposed to $\mathbb{R}$ for the real line) stands for an arbitary but fixed nontrivial boolean algebra of events [or sets] $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$, c, d,.. . When R is considered (via the Stone Representation Theorem or directly) to be such that $R \subseteq P(\Omega)$, for some set $\Omega \neq \emptyset$, the following can be interpreted alternatively via the bracketed quantities, where it is understood that $\emptyset, \Omega \in R$: conjunction [intersection $\cap$] is denoted by $\cdot$, or omitted altogether for simplicity when context is clear; disjunction [union $\cup$] is v; negation is $(\ )'$ [complement C or set difference $(\ ) - (\ )$]; + represents exclusive disjunction [symmetric set difference Δ] ; the special elements 0 [$\emptyset$], 1 [Ω] ε R denote the zero, unity elements, respectively. $\leqslant$ [$\subseteq$] is the natural partial order (a lattice order) over R^2 represented by the relation $a \leqslant b$ iff $a = a \cdot b$ iff $b = b \vee a$; material/logical implication is denoted $\Rightarrow$, where $b \Rightarrow a \overset{d}{=} b' \vee a = b' \vee a \cdot b = b' + a \cdot b$; material/logical equivalence is $\Leftrightarrow$, where $a \Leftrightarrow b \overset{d}{=} (b \Rightarrow a) \cdot (a \Rightarrow b) = ab \vee a'b' = (a+b)'$. Finally, typically, probability measures (assumed finitely additive or if needed, countably additive) are given in the form $p: R \to \mathcal{U}$, where $\mathcal{U} \overset{d}{=} [0,1]$ = the real unit interval.

Introduction

Conditional events have been developed in order to provide a systematic way to determine evaluations of arbitrary logical combinations of conditional or implicative statements with differing antecedents, so that each is consistent with conditional probability. Thus, when one seeks to obtain the probability of a compound statement such as "((if b then a) or (if d then not(c))) but not e)", traditional methods are inadequate in dealing with this. For example, if the well-known material implication is used to interpret the conditionals so that ordinary boolean algebra and properties of probability can be used for the full evaluation, before proceeding one should note that the probabilities do not match the corresponding conditional probability forms :

$$p(b \Rightarrow a) \neq p(a|b) \overset{d}{=} p(ab)/p(b)\ , \text{ for } p(b)>0, \quad (1)$$

and similarly for "if d then not(c)". In fact, it can be seen ([1],p.201) that

$$p(b \Rightarrow a) = 1-p(b)+p(ab) = p(a|b) + p(a'|b)p(b') \geq p(a|b), \quad (2)$$

with strict inequality holding in general. Going further, Calabrese([1],Th.2.2) showed no binary boolean function $g: R^2 \to R$ exists (of the 16 possible ones) for which

$$p(g(a,b)) = p(a|b),\ p(b)>0;\ \text{all } a,b \in R;\ \text{all } p: R \to \mathcal{U}. \quad (3)$$

Earlier, Lewis[2] had shown that g *not* satisfying (3) could be extended essentially to any binary function (not just boolean). (See also Goodman & Nguyen([3],ch.1) for a related result restricted to finite R, using a cardinality argument involving range(p).)

For a thorough history of both the negative results surrounding (2) and (3), as well as previous scattered attempts at constructing a satisfactor[illegible] for "conditional" events, see Goodman [4] o[illegible] Goo[illegible] & Nguyen [3]. Briefly, one should mention the original contribution of Boole ([5],ch.6+), Hailperin's rigorizing of Boole's attempts [6], DeFinetti's work [7], Schay's efforts [8], Adams' work([9], chp. II), and more recently, Calabrese [1] and Bruno & Gilio [10], among others. In all of the above, only DeFinetti and Schay considered conditional events through extensions of the usual indicator function, with only Schay developing a full conditional event algebra. Adams proposed extensions of the usual boolean operators to conditional forms, but did not give any real interpretation to what conditional events meant, nor did he investigate to the depth that Calabrese carried out in the latter's fully developed conditional event algebra.

Conditional Events Identified as Principal Ideal Cosets

In response to the previous unconnected efforts, Goodman [11] and Goodman & Nguyen [12],[13],[3] developed a fresh approach to conditional event algebra. Recall the basic concept of the principal ideal

TH0334-3/90/0000/0031$01.00 © 1990 IEEE

in R generated by any $b'\in R$ as $Rb' \stackrel{d}{=} \{xb' : x\in R\} \subseteq R$, leading to the boolean quotient algebra $R/Rb' = \{Rb'+a : a \in R\}$ with the usual well-defined coset operations for the cosets $Rb'+a \stackrel{d}{=} \{xb'+a : x\in R\}$ [14]. Denote the class of all such principal ideal cosets of R as $\tilde{R} \stackrel{d}{=} \cup\{R/Rb' : b\in R\}$ and the natural mapping $nat: R^2 \to \tilde{R}$, $nat(a,b) \stackrel{d}{=} Rb'+a = \{y : y\in R \;\&\; yb=ab\} = \{x : x\in R \;\&\; ab \leq x \leq b \Rightarrow a\}$. If $g: R^2 \to S$ is to be a reasonable candidate for a conditional event of the form $g(a,b)$, for any $a,b \in R$, then Lewis' result shows that at least $S \not\subseteq R$, when range$(g)=S$. In addition, one should assume that:

(i) Antecedent-consequent invariance

$$g(a,b)=g(ab,b), \text{ all } a,b \in R \tag{4}$$

(ii) Unique global representation

$$g(a,b)=g(c,d) \text{ implies } ab=cd \;\&\; b=d, \text{ all } a,b,c,d \in R. \tag{5}$$

(ii) allows for the definition

$$p(g(a,b))=p(a|b) \tag{6}$$

to be well-defined. Call any such g possessing properties (i) and (ii) a *feasible candidate for forming conditional events*. Then the following holds:

Theorem 1. Goodman & Nguyen [3], chp.2.
(i) nat is a feasible candidate for forming conditional events and for each fixed b, $nat(\cdot,b): R \to R/Rb'$ is a homomorphism wrt coset operations.
(ii) If $g: R^2 \to S$ is any feasible candidate for forming conditional events, then g is globally isomorphic to nat. That is, there exists bijection $\psi: S \to \tilde{R}$, where $\psi \circ g = nat$ and for each $b\in R$, ψ_b: range$(g(\cdot,b)) \to R/Rb'$ is a bijection, and hence an isomorphism for the usual induced operations over range$(g(\cdot,b))$ through R/Rb', where $\psi_b(g(a,b)) \stackrel{d}{=} nat(a,b)$. ∎

Remarks.
(i) The above theorem justifies naturally the choice of principal ideal cosets of R for its conditional events, so that one defines for all $a,b \in R$,

$$(a|b) \stackrel{d}{=} Rb'+ab, \quad (R|R) \stackrel{d}{=} \tilde{R} = \{(a|b) : a,b \in R\} \subseteq P(R). \tag{7}$$

(ii) Hailperin [6] approached the above identification from the Chevalley-Uzkov algebraic fraction viewpoint and obtained the same result, while Calabrese took a logical deduct approach ([1], sect.3), which was later shown also to be equivalent to (7)-[12], eqs.(2.19)-(2.25).
(iii) Special conditional events: *unconditional events* $a=(a|1)$, whence $R \subseteq (R|R)$; the *indeterminate conditional event* $(a|0)=(0|0)$; the *unity-type conditional events* $(1|b)=(b|b)=Rb' \vee b = R\vee b$ = principal filter generated by b in R, $b\neq 0$; the *zero-type conditional events* $(0|b)=(b'|b)=Rb'$ = principal ideal generated by b' in R, $b\neq 0$.
(iv) Note the relations for all $a,b,c,d \in R$:

$$1=(1|1); 0=(0|1) \;;\; (a|b)=(c|d) \text{ iff } ab=cd \;\&\; b=d. \tag{8}$$

Conditional Events Identified with Three-Valued Indicator Functions

DeFinetti [7] and, independently, Schay [8] extended the ordinary indicator functions of sets to three values to represent conditional events by the mapping $\phi: R \to \{0,u,1\}^{\Omega}$, assuming $R \subseteq P(\Omega)$, where wlog, we also assume here the third value is u (entire unit interval). From now on denote $\{0,u,1\}$ by Q_0. Then, for all $(a|b) \in (R|R)$ and all $\omega \in \Omega$,

$$\phi(a|b)(\omega) \stackrel{d}{=} \begin{cases} 1, & \text{if } \omega \in ab \\ 0, & \text{if } \omega \in a'b \\ u, & \text{if } \omega \in b' \end{cases} \tag{9}$$

The following theorem can help motivate the choice of operators over (R|R) extending the boolean ones over R, assuming R is atomic and noting $0 \leq u \leq 1$:

Theorem 2. Goodman & Nguyen [3], chp. 5.
Let $R \subseteq P(\Omega)$. For any $(a|b),(c|d) \in (R|R) - \{(0|0)\}$:
(i) When $(a|b)$ is not zero-type, $(c|d)$ not unity-type:
(I) $\phi(a|b) \leq \phi(c|d)$ pointwise over Ω,
(II) $ab \leq cd$ & $c'd \leq a'b$, i.e., $ab \leq cd$ & $b \Rightarrow a \leq d \Rightarrow c$,
(III) $p(a|b) \leq p(c|d)$, all prob. $p: R \to u$; $p(b), p(d) > 0$,

are all equivalent statements.

(ii) $(a|b)$ is of zero-type iff $\phi(a|b) \leq u$ over Ω iff $p(a|b)=0$, all prob. $p: R \to u$, $p(b)>0$.
(iii) $(c|d)$ is of unity type iff $u \leq \phi(c|d)$ over Ω $p(c|d)=1$, all prob. $p: R \to u$, $p(d)>0$. ∎

Remarks.
(i) The indeterminate element is the only $(a|b)$ for which $\phi(a|b)=u$ identically over Ω.
(ii) It is desirable to obtain a conditional event algebra of operations yielding a partial order over $(R|R)^2$ extending the unconditional counterpart $\leq$ over R^2, compatible with Theorem 2. This is seen to be the case as presented in the next section.

Functional Image Approach to Extending Boolean Operators over R to (R|R)

As mentioned before, Adams, Schay, and Calabrese have independently proposed extensions of boolean operators to (R|R), details of which will be shown later. These operators were based upon empirically appealing, but ad hoc, considerations. The thinking of Goodman & Nguyen has been, on the other hand to use the natural way one extends "point"-valued functions to set valued ones: $g: X \to Y$ extends by the well-known *functional image approach* to simply $\hat{g}: P(X) \to P(Y)$, via $\hat{g}(A) \stackrel{d}{=} \{g(x) : x\in A\}$, all $A \in P(X)$. Since $(R|R) \subseteq P(R)$, it seems reasonable to attempt to extend the ordinary boolean operators over R by the functional image approach restricted to (R|R), with the expectation that closure holds not just for $P(R)$ (trivially), but for (R|R) itself. This is indeed so:

Theorem 3. [12], [13].
For all $a,b,c,d,a_j,b_j \in R$, $j=1,\ldots,n$ arbitrary:
(i)

$$\begin{aligned} (a|b)' &\stackrel{d}{=} \{x' : x\in(a|b)\} = (a'|b), \\ (a|b)\cdot(c|d) &\stackrel{d}{=} \{x\cdot y : x\in(a|b), y\in(c|d)\} = (abcd|r_2), \\ (a|b)\vee(c|d) &\stackrel{d}{=} \{x\vee y : x\in(a|b), y\in(c|d)\} = (ab \vee cd|q_2), \\ (a|b)+(c|d) &\stackrel{d}{=} \{x+y : x\in(a|b), y\in(c|d)\} = (ab+cd|s_2), \end{aligned} \tag{10}$$

where

$$r_2 \stackrel{d}{=} a'b \vee c'd \vee abcd; \; q_2 \stackrel{d}{=} ab \vee cd \vee a'bc'd \;;\; s_2 \stackrel{d}{=} bd. \tag{11}$$

More generally, it can be shown

$$\begin{aligned} (a_1|b_1)\cdots(a_n|b_n) &= \Big(\mathop{\cdot}_{j=1}^{n} a_jb_j \Big| r_n\Big); \; r_n \stackrel{d}{=} \bigvee_{j=1}^{n} a'_jb_j \vee \mathop{\cdot}_{j=1}^{n} a_jb_j, \\ (a_1|b_1)\vee\ldots\vee(a_n|b_n) &= \Big(\bigvee_{j=1}^{n} a_jb_j \Big| q_n\Big); \; q_n \stackrel{d}{=} \bigvee_{j=1}^{n} a_jb_j \vee \mathop{\cdot}_{j=1}^{n} a'_jb_j, \\ (a_1|b_1)+\ldots+(a_n|b_n) &= \Big(\mathop{+}_{j=1}^{n} a_jb_j \Big| s_n\Big); \; s_n \stackrel{d}{=} \mathop{\cdot}_{j=1}^{n} b_j. \end{aligned} \tag{12}$$

(ii) Extend natural partial order $\leq$ over R^2 to $\leq$ over $(R|R)^2$, where by definition,

$$(a|b) \leq (c|d) \text{ iff } (a|b) = (a|b)\cdot(c|d). \tag{13}$$

Then,

$$(a|b) \leq (c|d) \text{ iff } (c|d)=(a|b)\vee(c|d) \text{ iff } ab \leq cd \;\&\; c'd \leq a'b. \tag{14}$$

(iii)

$$(a|b)\cdot b = ab; \; (a|bc)\cdot(c|b) = (ac|b); \text{ (chaining)} \tag{15}$$

If $\bigvee_{j=1}^{n} a_j \geq b$, $(a_j|b) = \Big((b|a_j)\cdot a_j \Big| \bigvee_{j=1}^{n} (b|a_j)\cdot a_j\Big)$, (Bayes' theorem) ∎

Remarks.

(i) Theorem 3 shows that any finite logical combination of logical connectors of conditional statements can be evaluated compatible with all probability evaluations, thus addressing the motivating problem for developing conditional event algebras.

(ii) Applying Theorem 3(ii) to Theorem 2 answers in the affirmative the remark, part (ii) following Theorem 2 : the extended lattice or partial order $\leq$ over $(R|R)^2$ yields the compatibility, for all $(a|b)$, $(c|d) \in (R|R)$ with $(a|b)$ not zero-type, $(c|d)$ not unity type:

$$\phi(a|b) \leq \phi(c|d) \text{ over } \Omega \text{ iff } (a|b) \leq (c|d) \text{ iff for all prob. } p: R \to \mathcal{U},\ p(b), p(d) > 0,\ p(a|b) \leq p(c|d). \quad (16)$$

(iii) A third type of justification for employing the conditional event algebra proposed here is provided by the next theorem, where it is seen:that this conditional event algebra has almost all the properties of a boolean algebra; that it can be categorized completely algebraically as a Stone algebra with some additional properties; and that it can be used to extend fully the Stone Representation Theorem for the classical case (as in [14]).

Theorem 4. [3],chp.4.

(i) Consider $(R|R)$ relative to the operations and relations introduced by the functional image approach for $\cdot, \vee, (\ )', +, \leq$, keeping in mind the special elements $0, 1, (0|0)$. Then, $(R|R)$ is a *Stone Algebra*: It is a bounded (wrt lattice order $\leq$ below by 0, above by 1) lattice wrt $\cdot, \vee$ (hence, associative, commutative, idempotent, and absorbing)which is distributive mutually for $\cdot, \vee$, and $(\cdot, \vee, (\ )')$ is a DeMorgan triple. In general, however, $(R|R)$ is not orthocomplemented: the leading candidate $(\ )'$ fails, since for example, $(a|b)\cdot(a|b)' = (0|b) \neq 0$, unless $b=1$. The pseudocomplement mapping $(\ )^*: (R|R) \to (R|R)$ exists, extending $(\ )'$ over R, and satisfying the Stone condition (17)

$$(a|b)^* \vee (a|b)^{**} = 1, \text{ all } (a|b) \in (R|R).$$

(I) More on the pseudocomplementation of $(R|R)$: Actually, $(R|R)$ is not only pseudocomplemented, but relatively pseudocomplemented, extending the well-known property that R is relatively pseudocomplemented with, for all $a, b \in R$,

$$b \rhd a \overset{d}{=} \vee\{x : x \in R \ \&\ xb \leq a\} = b \Rightarrow a \in R. \quad (18)$$

Specifically, for all $(a|b), (c|d) \in (R|R)$,

$$(c|d) \rhd (a|b) \overset{d}{=} \vee\{(x|y) : (x|y) \in (R|R) \ \&\ (x|y)\cdot(c|d) \leq (a|b)\}$$
$$= \lambda \vee (a|b) = (\lambda \vee ab \,|\, \lambda \vee b);\ \lambda \overset{d}{=} b'd' \vee c'd \quad (19)$$

From this,

$$(c|d)^* \overset{d}{=} (c|d) \rhd 0 = c'd \in R \subseteq (R|R). \quad (20)$$

(II) $()'$ is involutive for $(R|R)$ and with $(\ )^*$,

$$(a|b)^{*\prime} = (a|b)^{**} \ (= b \Rightarrow a);\ (0|0)'^{*} = 0. \quad (21)$$

Referring to Grätzer e.g. [15], the skeletal and dense sets of $(R|R)$ are, respectively,

$$(R|R)^* \overset{d}{=} \{(a|b)^* : (a|b) \in (R|R)\};\ D(R|R) \overset{d}{=} \ker((\ )^*) = (\ )^{*-1}(0),$$

yielding the relation - since $(R|R)^* = R$, (22)

$$D(R|R) = (R|R)^* \vee (0|0)\ (= \{(b|b) : b \in R\}). \quad (23)$$

In a related vein, note the relations for all $a, b \in R$:

$$(R|R) = R \vee R\cdot(0|0) \text{ via } (a|b) = ab \vee b'\cdot(0|0). \quad (24)$$

(ii) Conversely to the above results, replacing $(R|R)$ by any abstract algebraic system $\mathcal{S}$ which is a Stone algebra (involutive wrt its $()'$ operator) satisfying the compatibility conditions of (i)(II), then also $\mathcal{S}$ is isomorphic to $(R|R)$, where here $R = \mathcal{S}^*$, necessarily a boolean algebra. Call the mapping $h: \mathcal{S} \to (R|R)$. In turn, if the standard Stone Representation mapping is denoted as $m: R \to P(\Omega)$, for any boolean algebra R, an injective isomorphism,m, it can be shown that the mapping $(m|m): (R|R) \to (P(\Omega)|P(\Omega))$ is also an injective isomorphism, extending m, where

$$(m|m)(a|b) \overset{d}{=} (m(ab)|m(b)), \text{ all } (a|b) \in (R|R), \quad (25)$$

with $(R|R)$ assigned the conditional event algebra as in (i). Hence, the composition of mappings $(m|m)\circ h: \mathcal{S} \to (P(\Omega)|P(\Omega))$ is an injective isomorphism, providing a concrete representation forany such *abstract conditional event algebra.* ∎

Basic Isomorphism between All 3-Valued Truth-Functional Logics and All Boolean-Extended Conditional Event Algebras

In the last section a compact detailed structural analysis of the Goodman & Nguyen [abbreviated from now on as GN] conditional event algebra was given. Much remains to be analyzed for the other leading candidate conditional event algebras, including the independently considered, but commonly structured, proposal of Schay (alternate choice one of two proffered [8]), Adams' [9], and Calabrese [1] [abbreviated from now on as SAC],and another of Schay's (alternate choice two- see again [8])[abbreviated from now on as simply S]. However, Schay ([8],Theorem 5) has derived Stone-like representations for, in effect,both SAC and S, corresponding to part of Theorem 4(ii) above.

For completeness, the basic operators for SAC and S are given below, with appropriately subscripted letters for all $(a|b), (c|d) \in (R|R)$. Once more, it is emphasized that GN, SAC, and S all agree on the essential structure of $(R|R)$-sans any algebraic operations,other than the classical coset ones for each fixed antecedent principal ideal boolean quotient algebra of parent boolean algebra R:

$$(a|b)'^{SAC} = (a|b)'^{S} \overset{d}{=} (a'|b)\ (= (a|b)'^{GN}): \quad (26)$$

$$(a|b) \vee_{SAC} (c|d) \overset{d}{=} (ab \vee cd \,|\, b \vee d) \quad ; \quad (27)$$

$$(a|b)\cdot_{SAC}(c|d) \overset{d}{=} ((a|b)'^{SAC} \vee_{SAC} (c|d)'^{SAC})'^{SAC}$$
$$= ((b \Rightarrow a)\cdot(d \Rightarrow c) \,|\, b \vee d) = (abd' \vee b'cd \vee abcd \,|\, b \vee d), \quad (28)$$

a DeMorgan relation;

$$(a|b) \vee_S (c|d) \overset{d}{=} (ab \vee cd \,|\, bd); \quad (29)$$

$$(a|b)\cdot_S(c|d) \overset{d}{=} ((a|b)'^{S} \vee_S (c|d)'^{S})'^{S} = (abcd \,|\, bd), \quad (30)$$

also a DeMorgan relation.

In addition, recently, Dubois & Prade [16],[17] have expressed interest in the development of the candidate conditional event algebras. In [16], pp.1112, 1113 and [17], pp.31-34, they have pointed out that the following correspondences hold between the three basic candidates and certain three-valued logics (although this was previously also indicated in [18] in preliminary form),using an informal argument:

$$SAC \leftrightarrow Sob_3\ ;\ S \leftrightarrow B_3\ ;\ GN \leftrightarrow Ł_3\ , \quad (31)$$

where Sob_3 indicates Sobocinski's three-valued logic (see [19] or Rescher [20], pp. 70,342), B_3 is Bochvar's internal three-valued logic ([20],pp.29-34, 339), and $Ł_3$ is Lukasiewicz' three-valued logic([20] pp. 22-28 and 335).

In this section a general theorem will be fully derived which constructively establishes an isomorphism between any choice of three-valued truth functional logical operator and any extended boolean conditional event operator (for definition, see below). First, some additional notation for multiple variables, as well as other concepts must be introduced ($R \subseteq P(\Omega)$):
Let n be any positive integer and $a, b, a_j, b_j \in R$ arb: $\underline{a} \overset{d}{=} (a_1, \ldots, a_n), \underline{b} \overset{d}{=} (b_1, \ldots, b_n)$ $\underline{a} \cdot \underline{b} \overset{d}{=} (a_1 b_1, \ldots, a_n b_n) \in R^n$; $\cdot(\underline{a}) \overset{d}{=} \underset{j=1}{\overset{n}{\cdot}} a_j \overset{d}{=} a_1 \cdots a_n \in R$; $(\underline{a}|\underline{b}) \overset{d}{=} ((a_1|b_1), \ldots, (a_n|b_n)) \in (R|R)^n$; and extend the three-valued indicator function for any $\omega \in \Omega$ as

$$\phi(\underline{a}|\underline{b})(\omega) \overset{d}{=} (\phi(a_1|b_1), \ldots, \phi(a_n|b_n)) \in Q_0^n . \quad (32)$$

Define the mappings $w_i : (R|R) \to R$, $i \in Q_0$, by

$$w_1(a|b) \overset{d}{=} ab \ ; \ w_0(a|b) \overset{d}{=} a'b \ ; \ w_u(a\ b) \overset{d}{=} b' \ , \quad (33)$$

and extending this, for any $\underline{j} \overset{d}{=} (j_1, \ldots j_n) \in Q_0^n$,

$$w_{\underline{j}}(\underline{a}|\underline{b}) \overset{d}{=} (w_{j_1}(a_1|b_1), \ldots, w_{j_n}(a_n|b_n)) \in R^n . \quad (34)$$

Also,

$$\mathrm{bool}_n(R) \overset{d}{=} \{g : g : R^n \to R \text{ is a boolean function}\} \quad (35)$$

and for any pair $g_1, g_2 \in \mathrm{bool}_{2n}(R)$, define the *extended boolean function over* $(R|R)^n$, $(g_1|g_2) : (R|R)^n \to (R|R)$, where for any $(\underline{a}|\underline{b}) = (\underline{a} \cdot \underline{b}|\underline{b}) \in (R|R)^n$,

$$(g_1|g_2)(\underline{a}|\underline{b}) \overset{d}{=} (g_1(a \cdot b, b) \mid g_2(a \cdot b, b)). \quad (36)$$

Lemma 1.
(i) For any $(a|b) \in (R|R)$, $\{w_i(a|b) . i \in Q_0\}$ is a partitioning of Ω, and more generally, so is $\{\cdot(w_{\underline{j}}(\underline{a}|\underline{b})) : \underline{j} \in Q_0^n\} \overset{d}{=} w(\underline{a}|\underline{b})$ a partitioning of Ω.

(ii)

$$\phi(a|b)^{-1}(i) = w_i(a|b), \text{ all } i \in Q_0 ,$$

and more generally, for all $\underline{j} \in Q_0^n$,
$\phi(\underline{a}|\underline{b})^{-1}(\underline{j}) = \cdot(w_{\underline{j}}(\underline{a}|\underline{b}))$; i.e., $\phi(\underline{a}|\underline{b})(\omega) = \underline{j}$ iff $\omega \in \cdot(w_{\underline{j}}(\underline{a}|\underline{b}))$. ∎

Lemma 2. For each $g \in \mathrm{bool}_{2n}(R)$, there is a minimal classwise nonvacuous index set $J_g \subseteq Q_0^n$ such that

$$g(\underline{a}|\underline{b}) = \underset{\underline{j} \in J_g}{v}(\cdot(w_{\underline{j}}(\underline{a}|\underline{b})) ; \text{ all } (\underline{a}|\underline{b}) \in (R|R)^n . \quad (37)$$

Proof: Use normal disjunctive form for boolean functions. ∎

Theorem 5. Let $g : (R|R)^n \to (R|R)$ be arbitrary in $(\mathrm{bool}_n(R)|\mathrm{bool}_n(R)) \overset{d}{=} \{g : g = (g_1|g_2) \text{ ext.bool.over}(R|R)^n\}$ (38)
Then, there is a unique function $\psi(g) : Q_0^n \to Q_0$ such that for all $(\underline{a}|\underline{b}) \in (R|R)^n$, all $\omega \in \Omega$,

$$\phi(g(\underline{a}|\underline{b}))(\omega) = \psi(g)(\phi(\underline{a}|\underline{b})(\omega)). \quad (39)$$

Constructive Proof: Let $g = (g_1|g_2)$, $g_k \in \mathrm{bool}_{2n}(R)$. By Lemma 2, for each k, k=1,2, there is $J_{g_k} \subseteq Q_0^{2n}$, such that (37) holds with $n \to 2n, g \to g_k$. Then, applying the definition of ϕ in (9), and using the partitioning properties of $w(\underline{a}|\underline{b})$ from Lemma 1(i), for $\omega \in \Omega$,

$$\phi(g(\underline{a}|\underline{b}))(\omega) = i \ \text{ iff } \omega \in C_{g,i}(\underline{a}|\underline{b}) \ , \ i \in Q_0 , \quad (40)$$

where

$$C_{g,i}(\underline{a}|\underline{b}) \overset{d}{=} \underset{\underline{j} \in J_{g,i}}{v}(\cdot(w_{\underline{j}}(\underline{a}|\underline{b}))) \ , \ i \in Q_0 , \quad (41)$$

$$J_{g,1} \overset{d}{=} J_{g_1} \cap J_{g_2} ; \ J_{g,0} \overset{d}{=} J_{g_2} - J_{g_1} ; \ J_{g,u} \overset{d}{=} Q_0^n - J_{g_2} , \quad (42)$$

Note that, while $w(\underline{a}|\underline{b})$ is a partitioning of Ω, $\{C_{g,i}(\underline{a}|\underline{b}) : i \in Q_0\}$ is a partitioning of Ω and $\{J_{g,i} : i \in Q_0\}$ is a partitioning of Q_0^n.
Next, define $\psi(g) : Q_0 \to Q_0$ as follows: For any $\underline{j} \in Q_0^n$, from the above remarks, there is a unique i,

$$i \overset{d}{=} \psi(g)(\underline{j}) \in Q_0 \text{ with } \underline{j} \in J_{g,i} \ , \quad (43)$$

whence

$$\underline{j} \overset{d}{=} \phi(\underline{a}|\underline{b})(\omega) \in J_{g,\psi(g)(\underline{j})} \text{, for each } \omega \in \Omega. \quad (44)$$

Finally, with $(\underline{a}|\underline{b}) \in (R|R)^n$ fixed arb., for any $\omega \in \Omega$, choose $\underline{j}$ as in the left hand side of (44), followed by defining i as in (43),(44). Then, by Lemma 1(ii) applied to (44), $\omega \in \cdot(w_{\underline{j}}(\underline{a}|\underline{b}))$. In turn, applying this to (40),(41), taking into account (44) again, shows that for the above $(\underline{a}|\underline{b}), \omega, \underline{j}, i$,

$$\phi(g(\underline{a}|\underline{b}))(\omega) = i . \quad (45)$$

On the other hand, (43) and (44) state immediately that

$$\psi(g)(\phi(\underline{a}|\underline{b})(\omega)) = i . \quad (46)$$

Thus, (45) and (46) together show (39) holding. ∎

Theorem 6. Let $h : Q_0^n \to Q_0$ be any function. Then, there exists a unique function $\psi^{-1}(h) : (R|R)^n \to (R|R)$ in $(\mathrm{bool}_n(R)|\mathrm{bool}_n(R))$ such that for all $(\underline{a}|\underline{b}) \in (R|R)^n$ and all $\omega \in \Omega$,

$$\phi(\psi^{-1}(h)(\underline{a}|\underline{b}))(\omega) = h(\phi(\underline{a}|\underline{b})(\omega)). \quad (47)$$

Constructive Proof: First, for each $i \in Q_0$ and any $(\underline{a}|\underline{b}) \in (R|R)^n$, define, analogous to (41), replacing $J_{g,i}$ there by $h^{-1}(i) \subseteq Q_0^n$,

$$C_{h,i}(\underline{a}|\underline{b}) \overset{d}{=} \underset{\underline{j} \in h^{-1}(i)}{v}(\cdot(w_{\underline{j}}(\underline{a}|\underline{b}))) , i \in Q_0 . \quad (47)$$

Note that since $\{h^{-1}(i) : i \in Q_0\}$ is a partitioning of Q_0^n and $w(\underline{a}|\underline{b})$ is a partitioning of Ω, then $\{C_{h,i}(\underline{a}|\underline{b}) : i \in Q_0\}$ is a partitioning of Ω. In turn, define for any $(\underline{a}|\underline{b}) \in (R|R)^n$,

$$\psi^{-1}(h)(\underline{a}|\underline{b}) \overset{d}{=} (C_{h,1}(\underline{a}|\underline{b}) | C_{h,1}(\underline{a}|\underline{b}) \vee C_{h,0}(\underline{a}|\underline{b})). \quad (48)$$

Now, taking ϕ over (48), using the definition in (9), the above remarks show for all $(\underline{a}|\underline{b}) \in (R|R)^n$, all $\omega \in \Omega$, and all $i \in Q_0$:

$$\left.\begin{array}{l} \phi(\psi^{-1}(h)(\underline{a}|\underline{b}))(\omega) = i \ \text{ iff } \ \omega \in C_{h,i}(\underline{a}|\underline{b}) \text{ iff, by} \\ \text{(47), there is some (unique) } \underline{j} \in h^{-1}(i) \text{ such that} \\ \omega \in \cdot(w_{\underline{j}}(\underline{a}|\underline{b})) \ \text{ iff , using Lemma 1(ii), there is} \\ \text{some (unique) } \underline{j} \in Q_0^n \text{ so that } h(\underline{j}) = i, \ \phi(\underline{a}|\underline{b})(\omega) = \underline{j}. \end{array}\right\} (49)$$

(49) thus shows finally the desired result in (47). ∎

Corollary 1. Referring to Theorems 5 and 6:
(i) $\psi : (\mathrm{bool}_n(R)|\mathrm{bool}_n(R)) \to Q_0^{Q_0^n}$ is a bijection which makes any $g \in (\mathrm{bool}_n(R)|\mathrm{bool}_n(R))$ commutative with the three-valued indicator mapping $\phi : (R|R) \to Q_0$ in the sense

$$\phi \circ g = \psi(g) \dot{\circ} \phi , \quad (50)$$

i.e., for all $(\underline{a}|\underline{b}) \in (R|R)^n$, $\omega \in \Omega$, eq.(39) holds.
(ii) In a sense equivalent to (i), $\phi : (R|R) \to Q_0$ is an isomorphism relative to $(\mathrm{bool}_n(R)|\mathrm{bool}_n(R))$ over $(R|R)^n$ and $Q_0^{Q_0^n}$ over Q_0^n.
Proof: Direct result of combining Theorems 5 and 6. ∎

Remark. Corollary 1 shows that *all* algebraic properties of $(R|R)$ relative to $(\mathrm{bool}_n(R)|\mathrm{bool}_n(R))$ and Q_0 relative to $Q_0^{Q_0^n}$ must coincide! This can be useful in developing properties for conditional event algebras via three-valued logics and vice versa. The next sections show how Corollary 1 (or Theorems 5 or 6) can be used to compare and contrast properties for various candidate conditional event algebras in addition to the three discussed earlier.

Further Results Using the Basic Isomorphism

Example illustrating conditional event algebra operations converted to 3-valued logic.

As an example how the constructive proof in Theorem 5 can be used, consider again the operator $g = \cdot_{SAC}$ $(R|R)^2 \to (R|R)$ from eq.(28). Here, n=2, $(\underline{a}|\underline{b}) = ((a|b),(c|d))$, $g = (g_1|g_2)$, where

$$g_1(\underline{a}\cdot\underline{b},b) = abd' \vee b'cd \vee abcd = w_1(a|b)w_u(c|d) \vee w_u(a|b)w_1(c|d) \vee w_1(a|b)w_1(c|d),$$

whence

$$J_{g_1} = \{(1,u),(u,1),(1,1)\};$$

$$g_2(\underline{a}\cdot\underline{b},\underline{b}) = b \vee d = ab \vee a'b \vee cd \vee c'd$$
$$= w_1(a|b) \vee w_0(a|b) \vee w_1(c|d) \vee w_0(c|d)$$
$$= \bigvee_{i\in Q_0} w_1(a|b)w_i(c|d) \vee \bigvee_{i\in Q_0} w_0(a|b)w_i(c|d) \vee \bigvee_{i\in Q_0} w_i(a|b)w_1(c|d) \vee \bigvee_{i\in Q_0} w_i(a|b)w_0(c|d),$$

whence

$$J_{g_2} = \{(1,0),(1,u),(1,1),(0,0),(0,u),(0,1),(0,1),(u,1),(1,1),(0,0),(u,0),(1,0)\}$$
$$= Q_0^2 - \{(u,u)\}.$$

Then, from eq.(42),

$$J_{g,1} = J_{g_1} \cap J_{g_2} = \{(1,u),(u,1),(1,1)\},$$

$$J_{g,0} = J_{g_2} - J_{g_1} = \{(0,0),(0,u),(0,1),(u,0),(1,0)\},$$

$$J_{g,u} = Q_0^2 - J_{g_2} = \{(u,u)\}.$$

Thus, for all $\underline{j}=(j_1,j_2)\in Q_0^2$, $\psi(\cdot_{SAC})(\underline{j})=i$, if $\underline{j}\in J_{g,i}$:

$\psi(\cdot_{SAC})$	j_2: 0	u	1
j_1: 0	0	0	0
u	0	u	1
1	0	1	1

value i=0, for $\underline{j}\in J_{g,0}$
value i=u, for $\underline{j}\in J_{g,u}$
value i=1, for $\underline{j}\in J_{g,1}$

Figure 1. Partitioning of values for the 3-valued logic operator corresponding to $\cdot_{SAC}$ via procedure of Theorem 5.

Example illustrating 3-valued logic operators converted to conditional event algebra.

As an example how the constructive proof in Theorem 6 can be used, consider the three-valed logical operator given in figure 1. We will show how the original generating conditional event operator - in this case $\cdot_{SAC}$ - can be recovered, knowing only the entries in the table.

First, obtain from the table, denoted as 3-valued logical operator h (replacing $\psi(\cdot_{SAC})$,

$$h^{-1}(0)=\{(0,0),(0,u),(0,1),(u,0),(1,0)\};$$

$$h^{-1}(u)=\{(u,u)\};\ h^{-1}(1)=\{(1,u),(1,1),(u,1)\}.$$

Next, obtain for any $(\underline{a}|\underline{b}) \overset{d}{=} ((a|b),(c|d)) \in (R|R)^2$,

$$C_{h,0}(\underline{a}|\underline{b}) = w_0(a|b)w_0(c|d) \vee w_0(a|b)w_u(c|d) \vee w_0(a|b)w_1(c|d) \vee w_u(a|b)w_0(c|d) \vee w_1(a|b)w_0(c|d)$$
$$= a'bc'd \vee a'bd' \vee a'bcd \vee b'c'd \vee abc'd$$
$$= a'b \vee c'd,$$

$$C_{h,1}(\underline{a}|\underline{b}) = w_1(a|b)w_u(c|d) \vee w_1(a|b)w_1(c|d) \vee w_u(a|b)w_1(c|d)$$
$$= abd' \vee abcd \vee b'cd,$$

$(C_{h,u}(\underline{a}|\underline{b}) = w_u(a|b)w_u(c|d) = b'd'$ not needed)

Compute: $C_{h,0}(\underline{a}|\underline{b}) \vee C_{h,1}(\underline{a}|\underline{b}) =$

$$a'b \vee a'bcd \vee c'd \vee abc'd \vee abd' \vee abcd \vee b'cd$$
$$= a'b \vee ab(c'd \vee d' \vee cd) \vee c'd \vee (a'b \vee ab \vee b')cd$$
$$= a'b \vee ab \vee c'd \vee cd$$
$$= a'b \vee c'd.$$

Hence

$$\psi^{-1}(h)(\underline{a}|\underline{b}) = (C_{h,1}(a|b) \mid C_{h,1}(\underline{a}|\underline{b}) \vee C_{h,0}(\underline{a}|\underline{b}))$$
$$= (abd' \vee abcd \vee b'cd \mid b \vee d),$$

which of course checks with $\cdot_{SAC}$ in eq.(28).

Applications to Comparing/Contrasting Conditional Event Algebras

Using the procedure in the examples, one can verify rigorously Dubois & Prade's conclusions in (31):

Corollary 2. $\phi:(R|R)\to Q_0$ is an isomorphism relative to:

(i) SAC-conditional event algebra over $(R|R)^n$ and Sob_3 logic over Q_0^n.

(ii) S-conditional event algebra over $(R|R)^n$ and B_3 logic over Q_0^n.

(iii) GN-conditional event algebra over $(R|R)^n$ and $Ł_3$ logic over Q_0^n. ■

Next, consider a number of desirable properties that a conditional event algebra should possess. By use of the transfer technique above, in general it will be more convenient to analyze the candidate conditional event algebras for these properties via the three-valued logic form, rather than in the original form. However, these properties will be given in the latter form initially with a circle about the corresponding ordinary boolean operator to indicate the generic form:

Details are not required for the standard concepts of associativity, commutativity, and idempotence for $(\odot,\circledvee)$, involutiveness for $()^{\circledcirc}$, $(\odot,\circledvee,()^{\circledcirc})$ being orthocomplemented (i.e., law of excluded middle holds) or being a DeMorgan triple, or, finally, for $(\odot,\circledvee)$ being mutually distributive. In addition, define the following by the associated equations for all $(a|b),(c|d) \in (R|R)$:

monotonicity $\begin{cases} \phi((a|b)\odot(c|d)) \le \phi(a|b),\phi(c|d), \\ \phi((a|b)\circledvee(c|d)) \ge \phi(a|b),\phi(c|d) \end{cases}$

zero-unity $\begin{cases} 0\odot(a|b) = 0,\ 0\circledvee(a|b) = (a|b) \\ 1\circledvee(a|b) = 1,\ 1\odot(a|b) = (a|b) \end{cases}$

common antecedent homomorphism $\begin{cases} (a|b)\odot(c|b) = (ac|b), \\ (a|b)\circledvee(c|b) = (a\vee c|b) \end{cases}$

chaining 1: $(a|b)\odot b = ab$; 2: $(a|bc)\odot(c|b) = (ac|b)$

full lattice: $(a|b)=(a|b)\odot(c|d)$ iff $(c|d)=(a|b)\circledvee(c|d)$

full compatibility $\begin{cases} \phi(a|b)\le\phi(c|d) \text{ iff } p(a|b)\le p(c|d) \text{ iff} \\ (a|b)\circledleq(c|d), \text{defined by full lattice,} \\ \text{for all } p:R\to u \text{ prob., } p(b),p(d)>0, \\ (a|b) \text{ not zero-, } (c|d) \text{ not unity-types} \end{cases}$

logical entailment tautologically preserved	$(c\|d) \circledS (a\|b)$ is a unity-type event iff $\phi(c\|d) \le \phi(a\|b)$, where $\circledS$ is any extension of $\Rightarrow$ over R^2 to $(R\|R)^2$
logical equivalence tautologically preserved	$(c\|d) \circledcirc (a\|b)$ is a unity-type event iff $(c\|d)=(a\|b)$, where $\circledcirc$ extends $\Leftrightarrow$ over R^2 to $(R\|R)^2$
relatively pseudocomplemented: see eq.(19)	

The candidate conditional algebras to be compared relative to the above properties will not only include the three basic ones (SAC, S, GN) but will also include, for general interest, *all* possible commutative, monotonic, DeMorgan [cmD] systems with implication and logical equivalence being in the same formal relation as $\Rightarrow$ and $\Leftrightarrow$ relative to R^2. For the latter class, the transfer technique shows immediately that the truth table for all such systems must have its conjunction operator as:

Table 1. Possible cmD's.

$\psi(\odot)$	0	u	1
0	0	0	0
u	0	(0 or u)	(0 or u)
1	0	(0 or u)	1

In turn, Table 1 allows only four possible candidates satisfying the required constraints. These are all presented in Table 2 below:

Table 2. The 4 possible cmD conjunctions.

$\psi(\odot_1)$	0	u	1	$\psi(\odot_2)$	0	u	1	$\psi(\odot_3)$	0	u	1	$\psi(\odot_4)$	0	u	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
u	0	0	0	u	0	0	u	u	0	u	0	u	0	u	u
1	0	0	1	1	0	u	1	1	0	0	1	1	0	u	1

Clearly, the fourth subtable above is the same as $Ł_3$ conjunction, i.e., min, which already has been introduced as corresponding to GN. It should be also noted that the second subtable above corresponds to the important connector cop_0, the smallest possible *copula*, where, for all $s,t \in u$,

$$cop_0(s,t) \stackrel{d}{=} \max(s+t-1, 0) \quad (u \text{ '=' } \tfrac{1}{2}). \quad (51)$$

(See [21] for background.) cop_0 also plays a key role, where $(cop_0, cocop_0, \min, \max)$ forms the foundation for a Chang or MV algebra, where $cocop_0$ is the DeMorgan dual of cop_0 (and hence the maximal such one)

$$cocop_0(s,t) \stackrel{d}{=} \min(s+t, 1) \quad (u \text{ '=' } \tfrac{1}{2}). \quad (52)$$

(See [22], p. 473 et passim for further details.)

Also, for completeness, the 3-valued logical tables corresponding to the three leading candidates will now be displayed for the conjunction operators:

Table 3. Conjunctions for $Sob_3, B_3, Ł_3$.

$\&_{Sob_3}$	0	u	1	$\&_{B_3}$	0	u	1	$\&_{Ł_3}$	0	u	1
0	0	0	0	0	0	u	0	0	0	0	0
u	0	u	1	u	u	u	u	u	0	u	u
1	0	1	1	1	0	u	1	1	0	u	1

Applying the transfer procedure of the second example, yields the following conditional event algebra correspondences to the conjunction operators in Table 2, for all $(a|b),(c|d) \in (R|R)$ (the disjunction being just the DeMorgan dual):

$$(a|b)\cdot_1(c|d) = abcd\,, \quad (53)$$

a scalar quantity!

$$(a|b)\cdot_2(c|d) = (abcd \,|\, a'b \vee c'd \vee abcd \vee b'd') \quad (54)$$

$$(a|b)\cdot_3(c|d) = (abcd \,|\, b \vee d). \quad (55)$$

Finally, as a check with eq.(10),

$$(a|b)\cdot_4(c|d) = (abcd \,|\, a'b \vee c'd \vee abcd). \quad (56)$$

Thus, in summary, the candidate conditional event algebras considered are represented by their conjunction operators given in eqs.(28),(30),(10),and (53)-(55), while their corresponding 3-valued logical conjunction operators are given in Tables 3 and 2. All of this leads to the next table providing a comparisons and contrasts for the above 6 systems, again obtained via the transfer technique, based upon Theorems 5 and 6:

Table 4. Comparisons of properties for 6 candidate conditional event algebras.

Conditional Event Algebra / Properties	SAC	S	GN	cmD_1	cmD_2	cmD_3
$\odot,\oplus$ associative	YES	YES	YES	YES	YES	YES
$\odot,\oplus$ commutative	YES	YES	YES	YES	YES	YES
$\odot,\oplus$ idempotent	YES	YES	YES	NO	NO	YES
$(\ )^{\circledcirc}$ involutive	YES	YES	YES	YES	YES	YES
$\odot,\oplus,(\ )^{\circledcirc}$ orthocom	NO	NO	NO	YES	YES	NO
$\odot,\oplus,(\ )^{\circledcirc}$ DeMorgan	YES	YES	YES	YES	YES	YES
$\odot,\oplus$ mut. distrib.	NO	YES	YES	YES	YES	NO
monotonicity	NO	NO	YES	YES	YES	YES
zero-unity	NO	NO	YES	NO	YES	NO
com. ante. homomor.	YES	YES	YES	NO	NO	YES
chaining prop. 1	YES	YES	YES	YES	YES	YES
chaining prop. 2	YES	YES	YES	NO	YES	YES
full $\odot,\oplus$ lattice	NO	NO	YES	NO	YES	NO
full compatibil.	NO	NO	YES	NO	NO	NO
logical ent. pres.	YES †3	NO	†1	NO	NO	NO
logical equ. pres.	YES †3	NO	†2	NO	NO	NO
rel. pseudocompl.	NO	NO	YES	NO	YES	NO

†1 YES, only if the consequent of material implication $((c|d)' \vee (a|b))$, i.e., $c'd \vee ab$ (using eq.(10)), is used in place of the usual $Ł_3$ implication, which by applying Theorem 6 to [20],p.23 is in fact in the form $b'd' \vee ((c|d)' \vee (a|b))$.

†2 YES, only if the consequent of material (logical) equivalence $(((c|d)' \vee (a|b))\cdot((a|b)' \vee (c|d)) = (ab \Leftrightarrow cd \,|\, bd))$, i.e., $ab \Leftrightarrow cd$, is used in place of the usual $Ł_3$ equivalence, which by applying Theorem 6 to [20],p.23 is in the form $b'd' \vee (ab \Leftrightarrow cd \,|\, bd)$.

†3 The YES response of SAC and the partial YES of GN (see †1,†2 above) are due to a characterization that these are the only possible systems preserving logical entailment and logical equivalence tautologically. (See [3].)

Remarks.

(i) Table 4 can be used immediately, e.g., to characterize GN as that unique conditional event algebra which is an idempotent, mutually distributive cmD having the common antecedent homomorphism property.

(ii) Additional properties of GN can be found in [3] where higher order conditional events and their homomorphic reductions are considered, as well as development of a conditional probability logic of propositions and the issue of relating the classical assignment of conditional probability to conditional events as functional image extensions. Furthermore, relations are developed between conditional random variables and ce's (conditional events), as well as, between qualitative conditional probability and ce's with interpretations for their outcomes through ϕ.

(iii) In an alternative direction, McCarthy has developed a three-valued logic responsive to the spirit of flow diagrams "if then, else.." [23], which has been greatly expanded and analyzed by Guzman & Squier [24], relating to a Kleene regular extension of classical logic. However, none of this has been related to probability computations in the sense discussed in this paper. It is of some interest, however, to be able to convert this *non-commutative logic* into a conditional event algebra. In particular, the proposed conjunction operator is given by the table

Table 5.

$C_{\&}$	0	u	1
0	0	0	0
u	u	u	u
1	0	u	1

Using Theorem 6, it readily follows that

$$(a|b)\,\psi^{-1}(C_{\&})\,(c|d)=(abcd\,|\,(a'\vee d)b)\ ,\text{all } a,b,c,d\in R. \tag{57}$$

Acknowledgments

The author wishes to acknowledge the support of the line management at NOSC, especially, M.M. Kudurian, Head of Advanced Concepts and Development Branch, and J.A. Salzmann, Head of Ashore Command and Intelligence Centers Division. In addition, acknowledgments are given here for the direct support of the Office of Naval Research, Code 1133 (under the sponsorship of Dr. R. Wachter) and the NOSC Program for Research and Technology, Code 014.

References

[1] P. Calabrese, "An algebraic synthesis of the foundations of logic and probability", *Info. Sci. 42*, 1987, pp. 187-237.

[2] D. Lewis, "Probabilities of conditionals and conditional probabilities", *The Philos. Rev., LXXXV(3)*, July, 1976, pp. 297-315. This and related articles are reprinted in *Ifs*, W.L. Harper, R. Stalnaker, G. Pearce, eds., Dordrecht, Holland: D. Reidel Co., 1981.

[3] I.R. Goodman and H.T. Nguyen, *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, Amsterdam: North-Holland Press, accepted for publication, 1991.

[4] I.R. Goodman, "Evaluation of combinations of conditional information I: a history", accepted for publication in *Info. Sci.*, 1991.

[5] G. Boole, *An Investigation of the Laws of Thought*, London: Walton & Maberly, 1854. Reprinted by Dover Books, New York, 1951.

[6] T. Hailperin, *Boole's Logic and Probability, 2nd Ed.*, New York: North-Holland Press, 1986.

[7] B. DeFinetti, *Theory of Probability, Vol. 1*, pp. 139-140, *Vol. 2*, pp. 266-267, 322, New York: John Wiley & Sons, 1974. See also his article (orig. 1937) "Foresight: its logical laws, its subjective sources", in *Studies in Subjective Probability*, H.E. Kyburg and H.E. Smokler, eds., New York: John Wiley & Sons, 1964.

[8] G. Schay, "An algebra of conditional events", *J. Math. Analy. & Applic. 24(2)*, Nov., 1968, pp. 334-344.

[9] E. Adams, *The Logic of Conditionals*, Dordrecht, Holland: D. Reidel Co., 1980.

[10] G. Bruno and A. Gilio, "Confronto fra eventi condizionati di probabiliti nulli nell' inferenza statistica bayesiana", *Revis. mate. per sci. econ. e soc.* (Milan, Italy), *8(2)*, 1985, pp. 141-152.

[11] I.R. Goodman, "A measure-free approach to conditioning", *Proc. 3rd AAAI Wrkshp. Uncert. in AI*, Univ. of Wash., Seattle, July 10-12, 1987, pp. 270-277.

[12] I.R. Goodman and H.T. Nguyen, "Conditional objects and the modeling of uncertainties", in *Fuzzy Computing*, M.M. Gupta and T. Yamakawa, eds., New York: North-Holland Press, 1988, pp. 119-138.

[13] I.R. Goodman and H.T. Nguyen, "Foundations for an algebraic theory of conditioning", accepted for publication in *Fuzzy Sets & Sys.*, 1991.

[14] E. Mendelson, *Boolean Algebra and Switching Circuits*, Schaum Outline Series, New York: McGraw-Hill Co., 1970.

[15] G. Grätzer, *General Lattice Theory*, Basel: Birkhäuser Verlag, 1978.

[16] D. Dubois and H. Prade, "Measure-free conditioning, probability and non-monotonic reasoning, *Proc. 11th Inter. Joint Conf. AI*, Detriot, Mich., Aug. 20-25, 1989, pp.1110-1114.

[17] D. Dubois and H. Prade, "The logical view of conditioning and its application to possibility and evidence theories", *Int. J. Approx. Reason.*, 4, 1990, pp. 23-46.

[18] I.R. Goodman, "History and background of combining conditional events with differing antecedents", unpublished paper, part of minisymposium "Conditional event algebras and conditional probability computations", SIAM Annual Meeting, San Diego, July 21, 1989.

[19] B. Sobocinski, "Axiomatization of a partial system of three-valued calculus of propositions", *J. of Comp. Sys.1*,1952,pp. 23-55.

[20] N. Rescher, *Many-Valued Logic*, New York: McGraw-Hill Co., 1969.

[21] T. Hailperin, "Best possible inequalities for the probability of a logical function of events", *Amer. Math. Month. 72(1)*,1965, pp. 343-359.

[22] C.C. Chang, "Algebraic analysis of many valued logics", *Trans. Amer. Math. Soc. 88(2)*, July, 1958, pp. 467-490.

[23] J. McCarthy, "A basis for a mathematical theory of computation", in *Computer Programming and Formal Systems*, P. Braffort and D. Hirschberg, eds., Amsterdam: North-Holland Press, 1963, pp. 33-70.

[24] F. Guzman and C.C. Squier, "The algebra of conditional logic", *Algeb. Univer.*, 27, 1990, pp. 88-110.